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# On the Explicit Solution of a Time-Optimal Control Problem by Means of One-Sided Spline Approximation

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In [*Israel J. Math.* **10** (1971), 261–274] Schoenberg gives the explicit solution of a time-optimal control problem using the extremal properties of the so-called perfect $B$-splines. In this paper we consider a modification of Schoenberg's problem by taking into account the inertia of the moving particle. We give an explicit solution of the modified problem by means of one-sided spline approximations based on perfect $B$-splines. 

## 1. INTRODUCTION

In this paper we consider the following time-optimal control problem which is a modification of the problem solved by Schoenberg [7]; we take into account the inertia of the moving particle.

A particle $P$ moves on the $y$-axis and should reach the two points $y_0 = 0$ and $y_1 = L$ cyclically as fast as possible. The motion is controlled by some appropriate smoothness conditions on the velocities of different orders of the particle and by a restriction on the energy which is available per one cycle. Moreover, we allow the particle to glide over the points $y_0 = 0$ and $y_1 = L$, respectively, and to spend the same time used to go through $[0, L]$ to return and come to rest at $y_0 = 0$ resp. $y_1 = L$. This property of the inertia of the particle $P$ also makes sense in view of practical problems since reaching a prescribed point as fast as possible is often more relevant than reaching this point at rest (to switch a contact on/off, to open/shut a valve, etc.).

Before we start to solve the problem we reformulate it in a precise mathematical form.

PROBLEM. *Let $n \geqslant 2$ be given arbitrarily. A particle $P$ moves on the y-axis*



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*from $y_0 = 0$ to $y_1 = L$ such that its location function $S_n$ and all velocities of different orders,*

$$S_n^{(k)}, \qquad k = 1, 2, ..., n-2,$$

*are absolutely continuous. We assume that the particle $P$ starts from rest at $y_0 = 0$ at the time $t = 0$:*

$$S_n^{(k)}(0) = 0, \qquad k = 0, 1, ..., n-2,$$

*and that $P$ reaches the point $y_1 = L$ at the time $t = T/2$ with arbitrary velocities. During the time $T/2 < t < T$ the particle is above $y_1 = L$, retards, and returns to $y_1 = L$. At the time $t = T$ the particle $P$ reaches the point $y_1 = L$ for the second time coming to rest:*

$$S_n(T) = L, \qquad S_n^{(k)}(T) = 0, \; k = 1, 2, ..., n-2.$$

*After having rested at $y_1 = L$ for an arbitrary time the particle returns from $y_1 = L$ to $y_0 = 0$ such that its location function $s_n$ and all velocities of different orders,*

$$s_n^{(k)}, \qquad k = 1, 2, ..., n-2,$$

*are absolutely continuous. Resetting the clock at $t = T$ and counting backwards, $P$ starts from rest at $y_1 = L$ at the time $t = T$, i.e.,*

$$s_n(T) = L, \qquad s_n^{(k)}(T) = 0, \; k = 1, 2, ..., n-2.$$

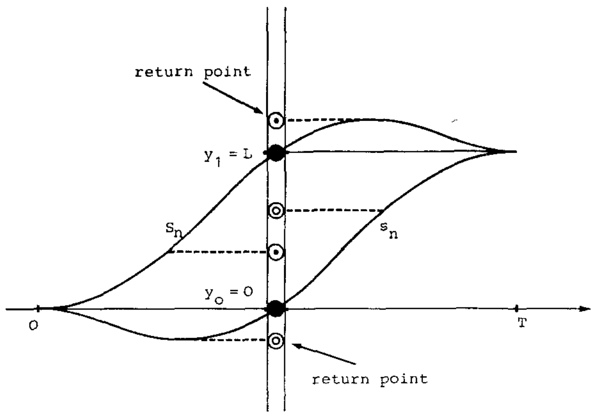


FIG. 1. ●, resting positions of $P$ of arbitrary duration; ⊙, $P$ on its way from $y_0 = 0$ to $y_1 = L$; ◎, $P$ on its way from $y_1 = L$ to $y_0 = 0$.

*Because of the symmetry of the problem P reaches the point $y_0 = 0$ at the time $t = T/2$ with arbitrary velocities. During the time $0 < t < T/2$ the particle is below $y_0 = 0$, retards, and returns to $y_0 = 0$. At the time $t = 0$ the particle P reaches the point $y_0 = 0$, again, coming to rest*:

$$s_n^{(k)}(0) = 0, \qquad k = 0, 1, ..., n-2.$$

*Now, we are interested in finding the shortest time 2T during which this motion can be performed and in describing the nature of this optimal motion with respect to the following additional restriction which may be interpreted as a property of the available energy*:

$$\|S_n^{(n-1)} - s_n^{(n-1)}\|_\infty \leqslant A.$$

*Here, $\|\cdot\|_\infty$ denotes the essential sup-norm with respect to $[0, T]$ and $A > 0$ is an arbitrarily given constant.*

## 2. Auxiliary Considerations Concerning One-Sided Spline Approximation

Analyzing the problem formulated in the first section we see that it is intimately connected with a special one-sided approximation problem of the step function $(x)_+^0$,

$$(x)_+^0 := \begin{Bmatrix} 1, & x \geqslant 0 \\ 0, & x < 0 \end{Bmatrix}.$$

In the following we will construct one-sided spline approximations of $(x)_+^0$ which will be used to solve the problem.

Let $-1 = x_0 < x_1 < \cdots < x_n = 1$, $n \geqslant 2$, be an arbitrary partition of $[-1, 1]$ and let

$$M_n(x) := n \sum_{i=0}^{n} \left\{ \prod_{\substack{j=0 \\ j \neq i}}^{n} (x_i - x_j) \right\}^{-1} (x_i - x)_+^{n-1} \tag{2.1}$$

with

$$(x_i - x)_+^{n-1} := \begin{cases} (x_i - x)^{n-1}, & x_i \geqslant x, \\ 0, & x_i < x, \end{cases} \quad 0 \leqslant i \leqslant n,$$

the corresponding $B$-spline of degree $n-1$. It is well known that

$$\begin{aligned} &(1) \quad M_n \in C^{n-2}(\mathbb{R}), \\ &(2) \quad M_n(x) \begin{cases} >0 & \text{for } |x|<1 \\ =0 & \text{for } |x| \geqslant 1, \end{cases} \\ &(3) \quad \int_{-\infty}^{\infty} M_n(x)\, dx = 1. \end{aligned}$$

Let us also consider the $B$-splines of degree $n-2$,

$$m_{n-1,0}(x) := (n-1) \sum_{i=0}^{n-1} \left\{ \prod_{\substack{j=0 \\ j \neq i}}^{n-1} (x_i - x_j) \right\}^{-1} (x_i - x)_+^{n-2} \tag{2.2}$$

and

$$m_{n-1,1}(x) := (n-1) \sum_{i=1}^{n} \left\{ \prod_{\substack{j=1 \\ j \neq i}}^{n} (x_i - x_j) \right\}^{-1} (x_i - x)_+^{n-2}, \tag{2.3}$$

with supports $[x_0, x_{n-1}] = [-1, x_{n-1}]$ and $[x_1, x_n] = [x_1, 1]$, respectively.

From now on we assume that the given partition of $[-1, 1]$ is symmetric, i.e.,

$$x_j = -x_{n-j}, \qquad 0 \leqslant j \leqslant n,$$

which implies that $\frac{1}{2}(m_{n-1,0} + m_{n-1,1})$ is an even function. Now, let us define

$$\varphi_n(x) := \int_{-\infty}^{x} \tfrac{1}{2}(m_{n-1,0}(t) + m_{n-1,1}(t))\, dt - \tfrac{1}{2}(M_n(0))^{-1} M_n(x), \tag{2.4}$$

$$\Phi_n(x) := \int_{-\infty}^{x} \tfrac{1}{2}(m_{n-1,0}(t) + m_{n-1,1}(t))\, dt + \tfrac{1}{2}(M_n(0))^{-1} M_n(x). \tag{2.5}$$

It is easy to check that

$$\begin{aligned} &(1) \quad \varphi_n(x) = \Phi_n(x) = 0 \qquad \text{for } x \leqslant -1, \\ &(2) \quad \varphi_n(x) = \Phi_n(x) = 1 \qquad \text{for } x \geqslant 1, \\ &(3) \quad \varphi_n(0) = 0, \\ &(4) \quad \Phi_n(0) = 1. \end{aligned}$$

Moreover, we have the following essential result.

LEMMA 2.1. *Let* $n \geqslant 2$ *and let* $-1 = x_0 < x_1 < \cdots < x_n = 1$ *be a symmetric partition of* $[-1, 1]$.

*Then the spline functions* $\varphi_n$ *and* $\Phi_n$ *are one-sided approximations of the step function* $(x)_+^0$, *i.e.*,

$$\varphi_n(x) \leqslant (x)_+^0 \leqslant \Phi_n(x), \qquad x \in \mathbb{R}. \tag{2.6}$$

*Especially, the corresponding one-sided approximation error has the representation*

$$\Phi_n(x) - \varphi_n(x) = (M_n(0))^{-1} M_n(x), \qquad x \in \mathbb{R}. \tag{2.7}$$

*Proof.* We only consider the left-hand inequality of (2.6) since the other may be proved by similar arguments.

In case $n = 2$ the knots are $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$. The interpolation conditions (1)–(4) stated above together with the fact that $\varphi_n$ is linear on $[-1, 0]$ and $[0, 1]$ immediately yield the desired inequality.

In case $n \geqslant 3$ $\varphi_n$ is at least one time continuously differentiable. Again, by means of the interpolation conditions (1)–(4) it is therefore sufficient to prove that $\varphi_n'$ has precisely one zero in $(-1, 1)$. Since

$$\varphi_n'(x) = \tfrac{1}{2}(m_{n-1,0}(x) + m_{n-1,1}(x)) \\ - \tfrac{1}{4}n(M_n(0))^{-1} (m_{n-1,0}(x) - m_{n-1,1}(x))$$

(cf. [8, p. 121, Theorem 4.16]) we obtain by means of Theorem 4.56, p. 162, of [8] that $\varphi_n'$ has at most—and therefore exactly—one zero in $(-1, 1)$. ∎

## 3. AUXILIARY RESULTS CONCERNING PERFECT B-SPLINES

In Section 2 we introduced the spline functions $\varphi_n$ and $\Phi_n$ which seem to be useful in solving the initial time-optimal control problem. Obviously, $\varphi_n$ and $\Phi_n$ satisfy the desired interpolation conditions of the problem, but—at the moment—nothing is known about the behaviour of $\Phi_n - \varphi_n$ in connection with the energy restriction. In the following we will show that the use of specially distributed knots leads to a solution of this aspect too.

From now on let

$$x_k := -\cos\frac{\pi k}{n}, \qquad k = 0, 1, ..., n,$$

be the extremal points of the $n$th Čebyšev polynomial and let $n \geqslant 2$ be arbitrary but fixed. The $B$-spline $M_n$ of degree $n-1$ corresponding to this partition is usually called the perfect $B$-spline of degree $n-1$.

Now, let us consider the following extremal interpolation problem (I):

*Consider the set $K_n$ of functions $f\colon [-1, 1] \to \mathbb{R}$ which have absolutely continuous derivatives $f^{(k)}$ for $0 \leqslant k \leqslant n-2$ and satisfy the symmetric interpolation conditions*

$$\begin{aligned} f^{(k)}(-1) &= 0, \qquad 0 \leqslant k \leqslant n-2, \\ f(0) &= 1, \\ f^{(k)}(1) &= 0, \qquad 0 \leqslant k \leqslant n-2. \end{aligned}$$

*Find a function $f_* \in K_n$ which satisfies*

$$\|f_*^{(n-1)}\|_\infty \leqslant \|f^{(n-1)}\|_\infty$$

*for all $f \in K_n$. Here, $\|\cdot\|_\infty$ denotes the essential sup-norm with respect to $[-1, 1]$.*

For the essential ideas used in finding the unique solution of problem (I) we refer the reader to [2, 3, 6, 7].

THEOREM 3.1. *Let $n \geqslant 2$ be given arbitrarily. The extremal interpolation problem* (I) *has a unique solution which is given by the perfect B-spline $M_n$ normalized with respect to the interpolation condition at $x = 0$, i.e.,*

$$f_*(x) = (M_n(0))^{-1} M_n(x). \tag{3.1}$$

*Moreover, the extremal derivative of $f_*$ satisfies the relation*

$$|f_*^{(n-1)}(x)| = (M_n(0))^{-1}\, 2^{n-2}(n-1)! \tag{3.2}$$

*for all $x \in (-1, 1) \backslash \{x_1, ..., x_{n-1}\}$.*

*Proof.* The fact that $f_*$ satisfies the interpolation conditions and the identity for $f_*^{(n-1)}$ is evident or even well known. Moreover, the results of [4, 1] immediately yield that $f_*$ is really a solution of problem (I). The only point in question is the uniqueness of $f_*$. In this direction we only have a result of Karlin [5, Theorem 6.2], which states that $f_*$ is the only perfect spline of degree $n-1$ with at most $(n-1)$ interior knots which solves problem (I). Now, we want to prove that $f_*$ is unique in $K_n$ also.

Let $f \in K_n$ be a function solving problem (I) and $t_1 \in (-1, 1)$, $t_1 \neq 0$, be a point with $f(t_1) \neq f_*(t_1)$. In the case of $t_1 \in (-1, 0)$ we may assume without loss of generality that there exists a point $t_2 \in (0, 1)$ with $f(t_2) \neq f_*(t_2)$, also, and vice versa. Since if there is no point in the other interval with different values of $f$ and $f_*$ we may only consider $\frac{1}{2}(f(x)+f(-x))$ instead of $f$ which is an extremal solution of problem (I) also. Taking into account the interpolation conditions

$$(f-f_*)(-1)=(f-f_*)(0)=(f-f_*)(1)=0$$

we therefore know that $f-f_*$ has at least two local extrema in $(-1, 0)$ and $(0, 1)$, respectively, with values different from zero. Now, using the extended version of Rolle's theorem $(n-2)$ times we obtain that $(f-f_*)^{(n-2)}$ has at least $(n-1)$ sign changes in $(-1, 1)$ (here and in the following an obvious modification of the manner of speaking is necessary in the case $n=2$). On the other hand $\|f^{(n-1)}\|_\infty = \|f_*^{(n-1)}\|_\infty$ together with the fact that $f_*^{(n-1)}$ is constant on each interval $(x_i, x_{i+1})$, $0 \leqslant i \leqslant n-1$, and taking alternating $\pm \|f_*^{(n-1)}\|_\infty$ imply that $(f-f_*)^{(n-2)}$ is monotone on each interval $(x_i, x_{i+1})$, $0 \leqslant i \leqslant n-1$. Since by means of the interpolation conditions we have

$$(f-f_*)^{(n-2)}(-1)=0=(f-f_*)^{(n-2)}(1)$$

the piecewise monotonicity of $(f-f_*)^{(n-2)}$ allows at most $(n-2)$ sign changes of $(f-f_*)^{(n-2)}$ which gives the desired contradiction. Therefore, $f(t)=f_*(t)$ for all $t \in [-1, 1]$ and $f_*$ is the unique solution of problem (I) in $K_n$. ∎

## 4. Solution of the Initial Time-Optimal Control Problem

According to the results obtained in Sections 2 and 3 we define the following functions leading—as we will show—to an optimal motion of the particle $P$,

$$s_{n,T}(t) := L\varphi_n\left(\frac{2t}{T}-1\right), \qquad t \in [0, T],\ T>0, \tag{4.1}$$

$$S_{n,T}(t) := L\Phi_n\left(\frac{2t}{T}-1\right), \qquad t \in [0, T],\ T>0. \tag{4.2}$$

Remembering the construction of $\varphi_n$ and $\Phi_n$ in Section 2 it is easy to check that $s_{n,T}$ and $S_{n,T}$ really describe a motion of $P$ in the sense of the initial control problem. Let us now—for a moment—fix $T>0$, arbitrarily, and consider any two functions $s_n$ and $S_n$ describing admissible motions of the particle $P$ of duration $2T$. Then, by change of scale the function $f$,

$$f(x) := \frac{1}{L}\left\{S_n\left(\frac{T}{2}(x+1)\right) - s_n\left(\frac{T}{2}(x+1)\right)\right\}, \qquad x \in [-1, 1],$$

belongs to the set $K_n$ of interpolation functions defined in Section 3. Since by Theorem 3.1

$$f_*(x) = (M_n(0))^{-1} M_n(x)$$

is optimal in $K_n$ with respect to the extremal condition and since

$$\frac{1}{L}\left\{S_{n,T}\left(\frac{T}{2}(x+1)\right) - s_{n,T}\left(\frac{T}{2}(x+1)\right)\right\} = f_*(x)$$

by Lemma 2.1 we obtain that $s_{n,T}$ and $S_{n,T}$ are optimal motions for each $T > 0$. Therefore, we get the minimal time $T^* > 0$ with respect to the given energy restriction by solving the identity

$$L\left(\frac{2}{T^*}\right)^{n-1} 2^{n-2}(n-1)!\,(M_n(0))^{-1} = A$$

which yields

$$T^* = 2\{L2^{n-2}(n-1)!\,(M_n(0)\cdot A)^{-1}\}^{1/(n-1)}. \tag{4.3}$$

In conclusion, we have the following result.

THEOREM 4.1. *Let $n \geqslant 2$ be given arbitrarily. Then the location functions*

$$s_{n,*}(t) := L\varphi_n\left(\frac{2t}{T^*} - 1\right), \qquad t \in [0, T^*] \tag{4.4}$$

$$S_{n,*}(t) := L\Phi_n\left(\frac{2t}{T^*} - 1\right), \qquad t \in [0, T^*] \tag{4.5}$$

*with*

$$T^* := 2\{L2^{n-2}(n-1)!\,(M_n(0)\cdot A)^{-1}\}^{1/(n-1)} \tag{4.6}$$

*are optimal solutions of the time-optimal control problem formulated in Section* 1.

*Moreover, this solution is unique in the following sense: If there are two other optimal location functions $r_{n,*}$ and $R_{n,*}$, then*

$$R_{n,*} - r_{n,*} = S_{n,*} - s_{n,*}. \tag{4.7}$$

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